

On Generalized Left Derivation on Semiprime Rings

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Received on: 29/12/2014 & Accepted on: 12/11/2015

ABSTRACT

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d , then R is commutative, if any one of the following conditions hold: (1) $[d(x), F(y)] = \pm[x, y]$, (2) $[d(x), F(y)] = \pm xoy$, (3) $d(x)oF(y) = \pm xoy$, (4) $d(x)oF(y) = \pm[x, y]$, for all $x, y \in R$.

Keywords: semiprime rings, prime rings, generalized left derivation, generalized Jordan left derivation.

حول الاشتقاق المعمم الأيسر على الحلقات شبه الأولية

الخلاصة:

لتكن R حلقة شبه أولية طليقة الالتواء من النمط 2، إذا احتوت R على الاشتقاق الأيسر المعمم F المرتبط باشتقاق جوردان الأيسر d ، فإن R حلقة تبادلية، إذا تحقق أحد الشروط التالية: (1) $[d(x), F(y)] = \pm[x, y]$ ، (2) $[d(x), F(y)] = \pm xoy$ ، (3) $d(x)oF(y) = \pm xoy$ ، (4) $d(x)oF(y) = \pm[x, y]$ ، لكل $x, y \in R$.

INTRODUCTION

Throughout, R will denote an associative ring with $Z(R)$. As usual, for any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy denotes for anti-commutator $xy + yx$. A ring R is n -torsion free, if $nx = 0, x \in R$ implies $x = 0$, where n is a positive integer. Recall that a ring R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$ and semiprime if $aRa = (0)$ implies $a = 0$. A prime ring is semiprime, but the converse is not true in general. An additive mapping $d: R \rightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Following [6] An additive mapping $d: R \rightarrow R$ is said to be left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in R$. Clearly, every left derivation on a ring R is a Jordan left derivation, but the converse need not to be true in general. A classical result of Ashraf and Rehman in [1] asserts that any Jordan left derivation on 2-torsion free prime ring is a left derivation.

In 1991, Brešar [5] defined the following concept. An additive mapping $F: R \rightarrow R$ is called a generalized derivation if there exists a derivation $d: R \rightarrow R$, such that:

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

And in [8] Jing and Lu, initiated the concept of generalized Jordan derivation as, an additive mapping $F: R \rightarrow R$ is called a generalized Jordan derivation if there exists a Jordan derivation $d: R \rightarrow R$, such that:

$$F(x^2) = F(x)x + xd(x), \text{ for all } x \in R.$$

Every generalized derivation is a generalized Jordan derivation, but the converse is not true in general. Jing and Lu in [8] proved that the converse is true when R is 2-torsion free prime ring.

Ashraf and Ali in [3] introduced the notion of generalized left derivation as follows: an additive mapping $F : R \longrightarrow R$ is called a generalized left derivation (resp. a generalized Jordan left derivation) associated with Jordan left derivation, if there exists a Jordan left derivation $d : R \longrightarrow R$, such that: $F(xy) = xF(y) + yd(x)$ (resp. $F(x^2) = xF(x) + xd(x)$) holds for all $x, y \in R$.

Posner [9], initiated the study of the commutativity of prime rings with derivation. Following, many authors proved the commutativity of prime rings with generalized derivation, for more details (see [2, 4, 10, and 11]). And in [3] Ashraf and Ali proved that, if R is a 2-torsion free prime ring, and R admits a generalized left derivation F associated with Jordan left derivation d , then either $d = 0$ or R is commutative.

In this paper we prove the commutativity of semiprime rings with generalized left derivation associated with Jordan left derivation, under certain conditions.

The Results

Theorem 1, [12]

Let R be a 2-torsion free semiprime ring with unity. Then every generalized Jordan left derivation F associated with Jordan left derivation d , is a generalized left derivation

The following theorem plays the key role in the proofs of the main results:

Theorem 2:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d , then $d(x) \in Z(R)$, for all $x \in R$.

For the proof of the theorem, we will need the lemma below:

Lemma 3:

Suppose that R is semiprime ring and $a \in R$, such that $a[a, x] = 0$, or $[a, x]a = 0$ for all $x \in R$, then $a \in Z(R)$.

Proof:

If $a[a, x] = 0$, for all $x \in R$, for the proof see [7].

Now, suppose:

$$[a, x]a = 0, \text{ for all } x \in R \dots\dots\dots(1)$$

Replace x by xr in (1) and using (1), we get:

$$[a, x]ra = 0, \text{ for all } x, r \in R \dots\dots\dots(2)$$

Replace r by rx in (2), to get:

$$[a, x]rxa = 0, \text{ for all } x, r \in R \dots\dots\dots(3)$$

Again, right multiplication of (2) by x , to get:

$$[a, x]rax = 0, \text{ for all } x, r \in R \dots\dots\dots(4)$$

Subtracting (3) from (4), we get:

$$[a, x]r[a, x] = 0, \text{ for all } x, r \in R \dots\dots\dots(5)$$

By semiprimeness of R , we get:

$$[a, x] = 0, \text{ for all } x \in R \dots\dots\dots(6)$$

And thus, $a \in Z(R)$. ■

Proof of Theorem 2:

We have:

$$F(x^2y) = x^2F(y) + yd(x^2), \text{ for all } x, y \in R \dots\dots\dots(1)$$

That is:

$$F(x^2y) = x^2F(y) + 2yxd(x), \text{ for all } x, y \in R \dots\dots\dots(2)$$

On the other hand, we find that:

$$F(x.xy) = xF(xy) + xyd(x), \text{ for all } x, y \in R \dots\dots\dots(3)$$

That is:

$$F(x^2y) = x^2F(y) + 2xyd(x), \text{ for all } x, y \in R \tag{4}$$

Comparing (2) and (4), we obtain:

$$2[x, y]d(x) = 0, \text{ for all } x, y \in R \tag{5}$$

Since R is 2-torsion free, we have:

$$[x, y]d(x) = 0, \text{ for all } x, y \in R \tag{6}$$

Linearizing (6) on x , we find that:

$$[x, y]d(w) + [w, y]d(x) = 0, \text{ for all } x, y, w \in R \tag{7}$$

Replacing y by yz in (6) and using (6), we have:

$$[x, y]zd(x) = 0, \text{ for all } x, y, z \in R \tag{8}$$

Replacing z by $d(w)z[w, y]$ in (8), we have:

$$[x, y]d(w)z[w, y]d(x) = 0, \text{ for all } x, y, z, w \in R \tag{9}$$

Comparing (7) and (9), we obtain:

$$[x, y]d(w)z[x, y]d(w) = 0, \text{ for all } x, y, z, w \in R \tag{10}$$

Since R is semiprime, we obtain:

$$[x, y]d(w) = 0, \text{ for all } x, y, w \in R \tag{11}$$

By Lemma 3, we get $d(w) \in Z(R)$, for all $w \in R$. ■

As an easy consequence of Theorem 2, we have:

Corollary 4:

Let R be a 2-torsion free semiprime ring, and $d: R \longrightarrow R$, then d is left derivation if and only if it is central derivation.

Corollary 5:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d , then $d(x) \in Z(R)$, for all $x \in R$.

Proof:

Using Theorem 1, we get R admits a generalized left derivation, and by **Theorem 2**, we get $d(x) \in Z(R)$, for all $x \in R$

Another immediate consequence of Theorem 2, is:

Corollary 6:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d , such that $[d(x), F(y)] = \pm[x, y]$, for all $x, y \in R$. Then R is commutative.

Corollary 7:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d , such that $[d(x), F(y)] = \pm[x, y]$, for all $x, y \in R$. Then R is commutative

Corollary 8:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d , such that $[d(x), F(y)] = \pm xoy$, for all $x, y \in R$. Then R is commutative.

Proof:

We assume:

$$[d(x), F(y)] = xoy, \text{ for all } x, y \in R \tag{1}$$

By Theorem (2), (1) gives:

$$xoy = 0, \text{ for all } x, y \in R \tag{2}$$

Replace y by yr in (2), to get:

$$xo(yr) = 0, \text{ for all } x, y, r \in R \tag{3}$$

This can be rewritten as:

$$(xoy)r - y[x, r] = 0, \text{ for all } x, y, r \in R \tag{4}$$

Using (2), (4) gives:

$$y[x, r] = 0, \text{ for all } x, y, r \in R \tag{5}$$

Left multiplication of (5) by $[x, r]$, and since R is semiprime, we get:

$$[x, r] = 0, \text{ for all } x, r \in R \tag{6}$$

Thereby, we get R is commutative.

Similarly, if $[d(x), F(y)] = -xoy$, for all $x, y \in R$, then R is commutative. ■

The last **Corollary**, led to:

Corollary 9:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d , such that $[d(x), F(y)] = \pm xoy$, for all $x, y \in R$. Then R is commutative.

Now, we will prove the main results:

Theorem 10:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d , such that $d(x)oF(y) = \pm xoy$, for all $x, y \in R$. Then R is commutative.

Proof:

We are given that:

$$d(x)oF(y) = xoy, \text{ for all } x, y \in R \tag{1}$$

By Theorem (2), (1) gives:

$$2d(x)F(y) = xoy, \text{ for all } x, y \in R \tag{2}$$

Replacing y by yz in (2), we obtain:

$$2d(x)yF(z) + 2d(x)zd(y) = xo(yz), \text{ for all } x, y, z \in R \tag{3}$$

Again, by Theorem (2), (3) gives:

$$2yd(x)F(z) + 2d(x)zd(y) = xo(yz), \text{ for all } x, y, z \in R \tag{4}$$

Since $xo(yz) = y(xoz) + [x, y]z$, then from (2) and (4), we obtain:

$$2d(x)zd(y) = [x, y]z, \text{ for all } x, y, z \in R \tag{5}$$

In particular, for $z = x$, (5) gives:

$$2d(x)xd(y) = [x, y]x, \text{ for all } x, y \in R \tag{6}$$

Replacing y by ry in (6), we obtain:

$$2d(x)xrd(y) + 2d(x)xyd(r) = [x, ry]x, \text{ for all } x, y, r \in R \tag{7}$$

Again, using Theorem (2), (7) reduces to:

$$2d(x)xd(y)r + 2d(x)xd(r)y = [x, ry]x, \text{ for all } x, y, r \in R \tag{8}$$

From (6) and (8), we obtain:

$$[x, y]xr + [x, r]xy = [x, ry]x, \text{ for all } x, y, r \in R \tag{9}$$

This implies that:

$$[[x, y]x, r] + [x, r][x, y] = 0, \text{ for all } x, y, r \in R \tag{10}$$

Replacing r by sr in (10), we obtain:

$$s[[x, y]x, r] + [[x, y]x, s]r + s[x, r][x, y] + [x, s]r[x, y] = 0, \text{ for all } x, y, r, s \in R \tag{11}$$

From (10) and (11), we obtain:

$$[[x, y]x, s]r + [x, s]r[x, y] = 0, \text{ for all } x, y, r, s \in R \tag{12}$$

Replacing r by rt in (12), we have:

$$[[x, y]x, s]rt + [x, s]rt[x, y] = 0, \text{ for all } x, y, r, s, t \in R \tag{13}$$

Again, right multiplying of (12) by t , we have:

$$[[x, y]x, s]rt + [x, s]r[x, y]t = 0, \text{ for all } x, y, r, s, t \in R \tag{14}$$

Subtracting (14) from (13), we obtain:

$$[x, s]r[t, [x, y]] = 0, \text{ for all } x, y, r, s, t \in R \tag{15}$$

Replacing y by yx in (15), we have:

$$[x, s]r[t, [x, y]x] = 0, \text{ for all } x, y, r, s, t \in R \tag{16}$$

Now, (10) can be rewritten as:

$$-[r, [x, y]x] + [x, r][x, y] = 0, \text{ for all } x, y, r \in R \tag{17}$$

For $r = t$ in (17), we have:

$$-[t, [x, y]x] + [x, t][x, y] = 0, \text{ for all } x, y, t \in R \tag{18}$$

Left multiplying of (18) by $[x, s]r$, we obtain:

$$-[x, s]r[t, [x, y]x] + [x, s]r[x, t][x, y] = 0, \text{ for all } x, y, r, s, t \in R \tag{19}$$

From (16) and (19), we obtain:

$$[x, s]r[x, t][x, y] = 0, \text{ for all } x, y, r, s, t \in R \tag{20}$$

Replacing r by $[x, y]r$ in (20), we have:

$$[x, s][x, y]r[x, t][x, y] = 0, \text{ for all } x, y, r, s, t \in R \tag{21}$$

For $s = t$ in (21) and since R is semiprime ring, we obtain:

$$[x, t][x, y] = 0, \text{ for all } x, y, t \in R \tag{22}$$

Replacing t by yt in (22) and using (22), we have:

$$[x, y]t[x, y] = 0, \text{ for all } x, y, t \in R \tag{23}$$

Invoking semiprimeness again, (23) gives:

$$[x, y] = 0, \text{ for all } x, y \in R \tag{24}$$

Thus R is commutative.

In the same way given $d(x) \circ F(y) = -xoy$, for all $x, y \in R$. The argument is similar. ■

Corollary 11:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d , such that $d(x) \circ F(y) = \pm xoy$, for all $x, y \in R$. Then R is commutative.

Theorem 12:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d , such that $d(x) \circ F(y) = \pm[x, y]$, for all $x, y \in R$. Then R is commutative.

Proof:

We assume:

$$d(x) \circ F(y) = [x, y], \text{ for all } x, y \in R \tag{1}$$

By Theorem (2.1), (1) gives:

$$2d(x)F(y) = [x, y], \text{ for all } x, y \in R \tag{2}$$

Replacing y by yz in (2) we obtain:

$$2d(x)yF(z) + 2d(x)zd(y) = y[x, z] + [x, y]z, \text{ for all } x, y, z \in R \tag{3}$$

Again, Theorem (2), (3) gives:

$$2yd(x)F(z) + 2d(x)zd(y) = y[x, z] + [x, y]z, \text{ for all } x, y, z \in R \tag{4}$$

From (2) and (4), we obtain:

$$2d(x)zd(y) = [x, y]z, \text{ for all } x, y, z \in R \tag{5}$$

In particular, (5) gives:

$$2d(x)xd(y) = [x, y]x, \text{ for all } x, y \in R \tag{6}$$

This implies (see how relation (24) was obtained from relation (6) in the proof of Theorem (10)) that R is commutative.

Similarly, we get R is commutative in case:

$$d(x) \circ F(y) = -[x, y], \text{ for all } x, y \in R. \quad \blacksquare$$

Corollary 13:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d , such that $d(x) \circ F(y) = \pm[x, y]$, for all $x, y \in R$. Then R is commutative.

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